MULTILINEAR PROCESSES IN BANACH SPACE

FRED ESPEN BENTH, NILS DETERING AND PAUL KRÜHNER

Abstract. A process $(X(t))_{t\geq 0}$ taking values in $\mathbb{R}^d$ is called a polynomial process if for every polynomial p of degree n on $\mathbb{R}^d$, there exists another polynomial q of degree at most n such that $\mathbb{E}[p(X(t))\,|\mathcal{F}_s] = q(X(s))$ for any $t \geq s \geq 0$. Based on multilinear maps we extend the notion of polynomial processes to a general Banach space B, to form a class of multilinear processes. If B is a Banach algebra and one restricts to multilinear maps being products, our notion of a multilinear process coincides with a canonical extension of polynomial processes from the finite dimensional case. While in a commutative Banach algebra, interesting examples of such polynomial processes exists, in a non-commutative Banach algebra multilinear maps arise naturally.

1. Introduction

Polynomial processes with values in the Euclidean space $\mathbb{R}^d, d < \infty$, or subsets thereof have received much attention recently especially due to their applications in financial mathematics. We refer the reader to Cuchiero, Keller-Ressel and Teichmann [9], Filipović and Larsson [13] and Foreman and Sørensen [14], and the references therein for an analysis and application of these processes and an overview of the existing literature. In the present paper we lift the notion of polynomial processes to general Banach spaces.

An $\mathbb{R}^d$-valued process $(X(t))_{t\geq 0}$ is said to be a *polynomial process* if for every polynomial p of degree n, there exists another polynomial q of degree at most n such that $\mathbb{E}[p(X(t))\,|\mathcal{F}_s] = q(X(s))$ for every $t \geq s \geq 0$. The polynomial q may have time-dependent coefficients, but not random. Examples of polynomial processes in $\mathbb{R}^d$ are affine processes or the multidimensional Jacobi process, among others (see Ackerer, Filipović and Pulido [1] for an application of the Jacobi process to stochastic volatility).

If $(X(t))_{t\geq 0}$ now takes values in some Banach algebra B instead of $\mathbb{R}^d$, the notion of polynomial process can be extended in a natural way as we show in this paper. Indeed, in a commutative Banach algebra, independent increment processes (among others) are polynomial in a canonical way based on the algebraic product operation. However, it turns out that in a non-commutative Banach algebra even independent increment processes are not polynomial and a more general structure is called for. We introduce multilinear mappings to extend the definition of polynomials. In fact, multilinear maps, and the corresponding definition of generalized

---

Date: September 3, 2018.

Key words and phrases. Infinite dimensional stochastic processes, polynomial processes, Banach algebras, conditional expectation, multilinear maps.

FEB acknowledges financial support from FINEWSTOCH, a research project funded by the Norwegian Research Council.

polynomials that we introduce, make sense in general Banach spaces without a designated multiplication operator.

To explain our approach in slightly more detail, let $\mathcal{L}_n : B^n \to B$, $n \in \mathbb{N}$, be a multilinear map on B^n, the product space of n copies of the Banach space B. We say that a B-valued stochastic process $(X(t))_{t\geq 0}$ is a *multilinear process* if for every $n \in \mathbb{N}$ and every multilinear map $\mathcal{L}_n$, it holds that

$$\mathbb{E}[\mathcal{L}_n(X(t), \ldots, X(t))\,|\mathcal{F}_s] = \sum_{k=0}^{n} \mathcal{M}_k(X(s;t), \ldots, X(s;t))$$

for all $t \geq s \geq 0$. Here, for $1 \leq k \leq n, \mathcal{M}_k$ are again multilinear maps, now on B^k and $\mathcal{M}_0 \in B$ is a constant. Further $X(s;t)$ is an $\mathcal{F}_s$ measurable random variable with values in B. Often this is simply $X(s)$, and moreover, the multilinear maps $\mathcal{M}_k$ may be depending (deterministically) on t and s. This of course includes the representation for monomials (and by linearity also polynomials) by defining $\mathcal{L}_n : B^n \to B$ by $\mathcal{L}_n(x_1, \ldots, x_n) = x_1 \cdots x_n$ if a designated multiplication in B is defined. The multilinearity property thus extends naturally the idea of the polynomial property in that *moment like* quantities of $(X(t))_{t\geq 0}$ can be easily calculated. The structure, however, does not focus on the particular moments arising from the designated multiplication operator.

Applications of our results in the setting of a commutative Banach algebra are for example commodity forward curve models in the Filipović space with the algebra defined by pointwise multiplication. Here we can exploit the polynomial structure in pricing of options on forwards. Additionally processes whose values are measures can be treated, linking our analysis and definitions to the work of Cuchiero, Larsson and Svaluto-Ferro [10]. In this case multiplication is the convolution product. Examples of relevance for non-commutative Banach algebras include the matrix-valued stochastic processes or more general processes of linear bounded operators where multiplication is the concatenation of operators. These cases cover infinite-dimensional stochastic volatility models (see Benth, Rüdiger and Süss [5]) and random matrices. The multilinear respective polynomial property of processes allows then in all the mentioned cases to calculate important quantities like conditional expected moments efficiently. By approximation more general conditional expectations can also be calculated.

The outline of the paper is as follows. Section 2 contains some important preliminary results about conditional expectation in Banach spaces and algebras. In Section 3 we introduce our notion of a multilinear process and prove our main results. In Section 4 we restrict the state space to be a commutative Banach algebra and analyse polynomial versus multilinear processes. We pay special attention to the Ornstein-Uhlenbeck dynamics. Finally, in Section 5 we provide several possible applications of our results.

2. Some preliminaries on conditional expectation in Banach spaces

Let B be a Banach space over a field $\mathbb{F}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. We denote the norm by $\|\cdot\|$ and by $\mathcal{B}(B)$ the Borel σ-algebra of B. Further, $L(B)$ denotes the space of bounded linear operators on B. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Following the usual terminology (see e.g. Def. 1.4 in van Neerven [16]), a B-valued random variable X is a mapping from Ω into B which is *strongly measurable*, that is, there exists a sequence of simple random variables

$X_n := \sum_{i=1}^n \mathbb{I}_{F_i} x_i$ where $F_i \in \mathcal{F}$, $x_i \in B, i = 1, \ldots, n, n \in \mathbb{N}$ such that $X_n \to X$ in B pointwise when $n \to \infty$. Here, we use the notation $\mathbb{I}_F : \Omega \to \{0, 1\}$ as the indicator function on a set $F \in \mathcal{F}$. If B is separable, then strong measurability is equivalent to measurability in the sense that $X^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{B}(B)$. As a consequence of the approximation $X_n \to X$ a random variable X takes values in the closed separable subspace $B_c := \overline{\texttt{span} \ \cup_{n \in \mathbb{N}} \texttt{ran}(X_n)}$, the closure of the subspace spanned by the ranges of the X_n.

In van Neerven [16], a mapping $X : \Omega \to B$ is said to be *strongly* $\mathbb{P}$*-measurable* if there exists a sequence of simple random variables X_n such that $X_n \to X$ in B $\mathbb{P} - a.s$ as $n \to \infty$. However, in view of Prop. 1.10 in van Neerven [16], there exists a version $\widetilde{X}$ which is strongly measurable of any strongly $\mathbb{P}$-measurable random variable X, and vice versa. Thus, in our analysis we will always choose the strongly measurable version of a random variable, and therefore stick to the notion of *strongly measurable* throughout.

A random variable X where $\mathbb{E}[\|X\|] < \infty$ is said to be Bochner integrable with respect to $\mathbb{P}$, and we define $\mathbb{E}[X]$ to be the Bochner integral (see e.g. Ch. 1§1 (J) of Dinculeanu [11])

$$\mathbb{E}[X] := \int_\Omega X \, d\mathbb{P}$$

Moreover, $\mathbb{E}[X] \in B$ and $\|\mathbb{E}[X]\| \leq \mathbb{E}[\|X\|]$. Given a σ-algebra $\mathcal{G} \subset \mathcal{F}$ and a Bochner integrable B-valued random variable, we define the conditional expectation $\mathbb{E}[X \,|\, \mathcal{G}]$ as the strongly $\mathcal{G}$-measurable random variable satisfying

$$\int_G \mathbb{E}[X \,|\, \mathcal{G}] \, d\mathbb{P} = \int_G X \, d\mathbb{P} \tag{1}$$

for all $G \in \mathcal{G}$ (see Def. 38 in Ch. 1§2 of Dinculeanu [11]). Thm. 50 and Prop. 37 in Ch. 1§2 of Dinculeanu [11] show that the conditional expectation exists and is unique $\mathbb{P} - a.s.$.

The next result shows that (conditional) expectation commutes with bounded linear operators:

Lemma 1. *Suppose that X is a B-valued random variable which is Bochner integrable and $\mathcal{L} \in L(B)$. Then $\mathcal{L}X$ is a B-valued random variable which is Bochner integrable, $\mathbb{E}[\mathcal{L}X] = \mathcal{L}\mathbb{E}[X]$ and*

$$\mathbb{E}[\mathcal{L}X \,|\, \mathcal{G}] = \mathcal{L}\mathbb{E}[X \,|\, \mathcal{G}]$$

for any $\mathcal{G} \subset \mathcal{F}$.

Proof. Since there exists a closed separable subspace $B_c \subseteq B$ with $\mathbb{P}[X \in B_c] = 1$, we know that also the range of $\mathcal{L}$ is separable. Now, the statement is given in Peszat and Zabczyk [17, Proposition 3.15(ii)]. □

As the following Lemma shows, if $f : B \to B$ is continuous, then $f(X)$ is strongly measurable whenever X is. This is stronger than the first claim in the Lemma 1 above. Indeed, the more general result holds:

Lemma 2. *If $f : B_1 \to B_2$ is a continuous map between two Banach spaces B_1 and B_2, then $f(X)$ is a strongly measurable B_2-valued random variable for any strongly measurable B_1-valued random variable X.*

Proof. For any sequence of simple random variables X_n converging to X, we have that $f(X_n)$ is converging to $f(X)$ by continuity. If $X_n = \sum_{i=1}^n \mathbb{I}_{F_i} x_i$ for disjoint sets $F_i \in \mathcal{F}$ (we can always do this by redefining the sum), we find $f(X_n) = \sum_{i=1}^n \mathbb{I}_{F_i} f(x_i)$, which shows that $(f(X_n))_{n\in\mathbb{N}}$ is a sequence of simple random variables converging pointwise to $f(X)$. Hence, $f(X)$ is strongly measurable. $\square$

Choosing $B = B_1$ and $B_2 = \mathbb{R}$, we find that $\|X\|$ is a measurable real-valued random variable when X is a B-valued strongly measurable random variable. This is true since $x \mapsto \|x\|$ is a continuous map, as the triangle inequality shows that $\|x\| \leq \|x-y\| + \|y\|$ and $\|y\| \leq \|x-y\| + \|x\|$, and therefore $|\|x\| - \|y\|| \leq \|x-y\|$.

Let us focus on the concept of independence for B-valued random variables. We recall that X is independent of a σ-algebra $\mathcal{G} \subset \mathcal{F}$ if the two sets $X^{-1}(A)$ and $G \in \mathcal{G}$ are independent for all $A \in \mathcal{B}(B)$ and $G \in \mathcal{G}$. If $f : B \to B$ is a measurable map, it follows that $f(X)$ is independent of $\mathcal{G}$ whenever X is independent of $\mathcal{G}$. This is so because for any $A \in \mathcal{B}(B)$, $(f(X))^{-1}(A) = X^{-1}(f^{-1}(A))$ and $f^{-1}(A) \in \mathcal{B}(B)$ as f is measurable. Moreover, if in addition f is continuous, we see from Lemma 2 that $f(X)$ is a strongly measurable random variable being independent of $\mathcal{G}$. As a particular case, assume that B is a Banach algebra, that is, B is equipped with a multiplication operator $\cdot : B \times B \to B$ such that $(B, +, \cdot)$ is an associative $\mathbb{F}$-algebra and $\|x \cdot y\| \leq \|x\|\|y\|$ for any $x, y \in B$. Consider the map $f : B \ni x \mapsto x^2 \in B$. By the norm property in a Banach algebra,

$$\|x^2 - y^2\| = \|x(x-y) + (x-y)y\| \leq (\|x\| + \|y\|)\|x-y\|,$$

it follows that f is continuous. Thus, we see that X^2 is independent of $\mathcal{G}$ whenever X is independent of $\mathcal{G}$. By induction, we have that X^k is independent of $\mathcal{G}$ whenever X is independent of $\mathcal{G}$ for any $k \in \mathbb{N}$.

We have the following result on conditional expectation of independent random variables:

Lemma 3. *If X is a B-valued Bochner-integrable random variable which is independent of the σ-algebra $\mathcal{G} \subset \mathcal{F}$, then*

$$\mathbb{E}[X \,|\, \mathcal{G}] = \mathbb{E}[X].$$

Proof. Since X is Bochner-integrable there is a separable closed subspace $B_c \subseteq B$ such that $\mathbb{P}[X \in B_c] = 1$. Consequently, we may assume that B is separable. The statement is given in Peszat and Zabczyk [17, Proposition 3.15(v)]. $\square$

In our analysis, a "freezing property" of conditional expectation will become important. To this end, we equip the product space $B \times B$ of the Banach space B with the max-norm, i.e., for any $x = (x_1, x_2) \in B \times B$, $\|x\|_2 := \max_{i=1,2} \|x_i\|$. Then, $B \times B$ is a Banach space again.

Proposition 4. *Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra and X a B-valued random variable independent of $\mathcal{G}$. Let further Y be a B-valued random variable which is strongly $\mathcal{G}$-measurable and $f : B \times B \to B$ continuous. Assume $f(X, y)$ is Bochner integrable for every $y \in B$ and $y \mapsto \mathbb{E}[f(X, y)]$ is continuous, and moreover that there exist positive $\mathbb{R}$-valued random variables $Z, \widetilde{Z}$ with $\mathbb{E}[Z] < \infty, \mathbb{E}[\widetilde{Z}] < \infty$ and*

$$\|\mathbb{E}[f(X,y)]_{y=\widetilde{Y}}\| \leq Z \tag{2}$$

$$\|f(X, \widetilde{Y})\| \leq \tilde{Z}, \tag{3}$$

for every B-valued random variable $\widetilde{Y}$ such that $\|\widetilde{Y}\| \leq \|Y\|$ a.s., then

$$\mathbb{E}[f(X,Y)\,|\,\mathcal{G}] = \mathbb{E}[f(X,y)]_{y=Y}.$$

Proof. First, by Lemma 2, we note that continuity of f implies that $f(X,Y)$ and $f(X,y)$ are strongly measurable for any $y \in B$ (using $B_2 = B$ and $B_1 = B \times B$ or $B_2 = B$). Choosing $\widetilde{Y} = Y$ in (3), we find that $\mathbb{E}[\|f(X,Y)\|] \leq \mathbb{E}[\widetilde{Z}] < \infty$, and therefore $f(X,Y)$ is Bochner integrable.

We need to show that $\mathbb{E}[\mathbb{I}_G f(X,Y)] = \mathbb{E}[\mathbb{I}_G \mathbb{E}[f(X,y)]_{y=Y}]$ for any $G \in \mathcal{G}$. For this, let $G \in \mathcal{G}$ and recall that since Y is strongly $\mathcal{G}$-measurable, there exists a sequence of $\mathcal{G}$-simple random variables $Y_n = \sum_{i=1}^n y_i \mathbb{I}_{A_i}$ with $y_i \in B$ and $A_i \in \mathcal{G}$ such that $Y_n \to Y$ pointwise. Moreover, by Thm. 6 in Ch. 1§1 C of Dinculeanu [11] we can choose Y_n such that $\|Y_n\| \leq \|Y\|$. We also notice that the we can select the sets $A_1, \ldots, A_n$ to be disjoint, as we do. We see that

$$f(X,Y_n) = \sum_{i=1}^n \mathbb{I}_{A_i} f(X,y_i)$$

and by assumption (3), we calculate

$$\begin{aligned}
\infty > \mathbb{E}[\|f(X,Y_n)\|] &= \mathbb{E}[\sum_{i=1}^n \mathbb{I}_{A_i} \|f(X,y_i)\|] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_{A_i} \|f(X,y_i)\|] \\
&= \sum_{i=1}^n \mathbb{P}(A_i)\mathbb{E}[\|f(X,y_i)\|].
\end{aligned}$$

In the last equality we used the fact that $\|f(X,y_i)\|$ is independent on $\mathcal{G}$, as f and $\|\cdot\|$ are continuous functions and X is independent of $\mathcal{G}$ by assumption. In particular, this shows that $\mathbb{E}[\|f(X,y_i)\|] < \infty$, and hence $f(X,y_i)$ is Bochner integrable. Therefore, by Lemma 3 it follows that

$$\mathbb{E}[f(X,y_i)] = \mathbb{E}[f(X,y_i)\,|\,\mathcal{G}] \tag{4}$$

On the other hand,

$$\mathbb{E}[f(X,y)]_{y=Y_n} = \sum_{i=1}^n \mathbb{I}_{A_i}\mathbb{E}[f(X,y_i)].$$

Hence, $\mathbb{E}[f(X,y)]_{y=Y_n}$ is strongly $\mathcal{G}$-measurable and Bochner integrable since, from norm inequality of Bochner integrals and assumption (3)

$$\begin{aligned}
\mathbb{E}\left[\|\mathbb{E}[f(X,y)]_{y=Y_n}\|\right] &= \mathbb{E}\left[\sum_{i=1}^n \mathbb{I}_{A_i}\|\mathbb{E}[f(X,y_i)]\|\right] \\
&= \sum_{i=1}^n \mathbb{P}(A_i)\|\mathbb{E}[f(X,y_i)]\| \\
&\leq \sum_{i=1}^n \mathbb{P}(A_i)\mathbb{E}[\|f(X,y_i)\|] \\
&= \mathbb{E}[\|f(X,Y_n)\|] < \infty.
\end{aligned}$$

Thus, we calculate

$$\begin{aligned}\mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y_n}] &= \mathbb{E}\left[\sum_{i=1}^n \mathbb{I}_G\mathbb{I}_{A_i}\mathbb{E}[f(X,y_i)]\right]\\ &= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_{G\cap A_i}\mathbb{E}[f(X,y_i)]]\\ &= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_{G\cap A_i} f(X,y_i)]\\ &= \mathbb{E}[\mathbb{I}_G f(X,Y_n)],\end{aligned}$$

where the third equality uses (4) and the defining properties of conditional expectation. To see that in fact $\mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y}] = \mathbb{E}[\mathbb{I}_G f(X,Y)]$ we need to show that $\lim_{n\to\infty}\mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y_n}] = \mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y}]$ and $\lim_{n\to\infty}\mathbb{E}[\mathbb{I}_G f(X,Y_n)] = \mathbb{E}[\mathbb{I}_G f(X,Y)]$. For this note that since f is continuous, $\mathbb{I}_G f(X,Y_n) \to \mathbb{I}_G f(X,Y)$ when $n\to\infty$. As $\|Y_n\| \le \|Y\|$, we have

$$\|\mathbb{I}_G f(X,Y_n)\| \le \mathbb{I}_G\|f(X,Y_n)\| \le \tilde{Z}$$

and thus from dominated convergence it follows that

$$\lim_{n\to\infty}\mathbb{E}[\mathbb{I}_G f(X,Y_n)] = \mathbb{E}[\mathbb{I}_G f(X,Y)]$$

For the other limit, we have from the continuity assumption on $y\mapsto\mathbb{E}[f(X,y)]$ that $\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y_n} \to \mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y}$ pointwise in B. Furthermore, by assumption (2)

$$\|\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y_n}\| \le \mathbb{I}_G\|\mathbb{E}[f(X,y)]_{y=Y_n}\| \le Z$$

Then, by dominated convergence, we obtain

$$\lim_{n\to\infty}\mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y_n}] = \mathbb{E}[\mathbb{I}_G\mathbb{E}[f(X,y)]_{y=Y}],$$

and the proposition follows. □

Remark that condition (2) is only used once, to obtain a uniform bound on $\|\mathbb{E}[f(X,y)]_{y=Y_n}\|$. Further, the continuity assumption on the function $y\mapsto\mathbb{E}[f(X,y)]$ is only used to have pointwise convergence. Both are used only in connection with concluding the final limit in the proof above.

In the remainder of this section we will focus on the case when B is a Banach algebra, and in particular show the following fundamental property for conditional expectation: Let $\mathcal{G}\subset\mathcal{F}$ be a σ-algebra and Y a $\mathcal{G}$-strongly measurable B-valued random variable, then

$$\mathbb{E}[YX\,|\,\mathcal{G}] = Y\mathbb{E}[X\,|\,\mathcal{G}]\,,\mathbb{E}[XY\,|\,\mathcal{G}] = \mathbb{E}[X\,|\,\mathcal{G}]Y$$

where X is a B-valued random variable, with X, YX and XY such that the conditional expectations are well-defined.

First, we show a Lemma which will become convenient:

Lemma 5. *Let (S,Σ,μ) be a measure space and B a Banach algebra. Suppose $F:(S,\Sigma,\mu)\to B$ is μ-integrable (that is, Bochner integrable with respect to μ) and $g\in B$. Then gF and Fg are μ-integrable, and*

$$\int_S gF(s)\,\mu(ds) = g\int_S F(s)\,\mu(ds) \qquad \int_S F(s)g\,\mu(ds) = \int_S F(s)\,\mu(ds)g.$$

Proof. Define the continuous linear maps

$$\mathcal{L}_g : B \to B, b \mapsto gb,$$
$$\mathcal{R}_g : B \to B, b \mapsto bg.$$

Then we find

$$\int_S F(s) g\mu(ds) = \int_S \mathcal{R}_g F(s)\mu(ds) = \mathcal{R}_g \left(\int_S \mathcal{F}(s)\mu(ds) \right) = \int_S \mathcal{F}(s)\mu(ds) g.$$

Similar with $\mathcal{L}_g$. □

The next Lemma shows that measurability is preserved under the product operation in the Banach algebra:

Lemma 6. *Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra, and suppose that Y and Z are two strongly $\mathcal{G}$-measurable B-valued random variables and that B is a Banach algebra. Then YZ and ZY are strongly $\mathcal{G}$-measurable B-valued random variables.*

Proof. The pair (Y, Z) is strongly $B \times B$-measurable and the multiplication η on B is a continuous map from $B \times B$ to B. Thus, Lemma 2 yields that $YZ = \eta(Y, Z)$ is strongly $\mathcal{G}$-measurable. □

We come to our final result of this section:

Proposition 7. *Let B be a Banach algebra and $\mathcal{G} \subset \mathcal{F}$ a σ-algebra. Assume that X and Y are two B-valued random variables where Y is strongly $\mathcal{G}$-measurable, X is Bochner-integrable and $\|X\|\|Y\|$ is $\mathbb{P}$-integrable. Then*

$$\mathbb{E}[YX \,|\, \mathcal{G}] = Y\mathbb{E}[X \,|\, \mathcal{G}], \qquad \mathbb{E}[XY \,|\, \mathcal{G}] = \mathbb{E}[X \,|\, \mathcal{G}]Y.$$

Proof. First, $\|XY\| \leq \|X\|\|Y\|$ and $\|YX\| \leq \|Y\|\|X\|$, so both XY and YX are Bochner integrable, and moreover, the conditional expectations of XY and YX with respect to $\mathcal{G}$ are well-defined. By assumption, the conditional expectation of X with respect to $\mathcal{G}$ is also well-defined.

By definition of the conditional expectation, $\mathbb{E}[X \,|\, \mathcal{G}]$ is strongly $\mathcal{G}$-measurable, and thus by Lemma 6, $Y\mathbb{E}[X \,|\, \mathcal{G}]$ is $\mathcal{G}$-strongly measurable. Let $Y_n = \sum_{i=1}^n y_i \mathbb{I}_{A_i}$ with $y_i \in B$ and $A_i \in \mathcal{G}$ for $i = 1, \ldots, n$ be a sequence of $\mathcal{G}$-simple random variables such that $Y_n \to Y$ pointwise and by Thm. 6 in Ch. 1§1 C of Dinculeanu [11], $\|Y_n\| \leq \|Y\|$. Let $G \in \mathcal{G}$ be an arbitrary set. We find

$$\begin{aligned}
\mathbb{E}[\mathbb{I}_G Y_n X] &= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_G \mathbb{I}_{A_i} y_i X] \\
&= \sum_{i=1}^n y_i \mathbb{E}[\mathbb{I}_{G \cap A_i} X] \\
&= \sum_{i=1}^n y_i \mathbb{E}[\mathbb{I}_{G \cap A_i} \mathbb{E}[X \,|\, \mathcal{G}]] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_G \mathbb{I}_{A_i} y_i \mathbb{E}[X \,|\, \mathcal{G}]] \\
&= \mathbb{E}[\mathbb{I}_G Y_n \mathbb{E}[X \,|\, \mathcal{G}]]
\end{aligned}$$

In the second and fourth equalities we applied Lemma 5, and in the third the definition of the conditional expectation. Now,

$$\|\mathbb{I}_G Y_n X\| \leq \mathbb{I}_G \|Y_n\| \|X\| \leq \|Y\| \|X\| \in L^1(P)$$

by assumption. Thus, as $Y_n X \to YX$, it follows by dominated convergence that $\mathbb{E}[\mathbb{I}_G Y_n X] \to \mathbb{E}[\mathbb{I}_G YX]$. On the other hand,

$$\|\mathbb{I}_G Y_n \mathbb{E}[X \,|\, \mathcal{G}]\| \leq \mathbb{I}_G \|Y_n\| \mathbb{E}[\|X\| \,|\, \mathcal{G}] \leq \|Y\| \mathbb{E}[\|X\| \,|\, \mathcal{G}]\,.$$

$a.s.$, by Jensen's inequality (Property 44 in Ch.1§2 H of Dinculeanu [11]). As Y is strongly $\mathcal{G}$-measurable, it follows from Lemma 2 that $\|Y\|$ is a real-valued $\mathcal{G}$-measurable random variable. From the properties of conditional expectation for real-valued random variables

$$\mathbb{E}[\|Y\| \mathbb{E}[\|X\| \,|\, \mathcal{G}]] = \mathbb{E}[\mathbb{E}[\|Y\| \|X\| \,|\, \mathcal{G}]] = \mathbb{E}[\|Y\| \|X\|] < \infty$$

by assumption. Hence, $\|Y\| \mathbb{E}[\|X\| \,|\, \mathcal{G}]$ is P-integrable, and since obviously it holds pointwise that $\mathbb{I}_G Y_n \mathbb{E}[X \,|\, \mathcal{G}] \to \mathbb{I}_G Y \mathbb{E}[X \,|\, \mathcal{G}]$ we find by dominated convergence that $\mathbb{E}[\mathbb{I}_G Y_n \mathbb{E}[X \,|\, \mathcal{G}]] \to \mathbb{E}[\mathbb{I}_G Y \mathbb{E}[X \,|\, \mathcal{G}]]$. We can therefore conclude

$$\mathbb{E}[\mathbb{I}_G Y \mathbb{E}[X \,|\, \mathcal{G}]] = \lim_{n\to\infty} \mathbb{E}[\mathbb{I}_G Y_n \mathbb{E}[X \,|\, \mathcal{G}]] = \lim_{n\to\infty} \mathbb{E}[\mathbb{I}_G Y_n X] = \mathbb{E}[\mathbb{I}_G YX].$$

Hence, the first result of the Proposition is proven. The second part follows in the same manner. □

The above results for conditional expectation in general Banach spaces will be used in our study of multilinear maps and processes, that follows next.

3. Multilinear maps and multilinear processes

In this section we study stochastic processes with values in B which possess certain stability properties with respect to "polynomials" and conditional expectation. We introduce polynomials via certain multilinear maps, that are defined next:

Denote by $B^k = B \times \cdots \times B$ the product space of $k \in \mathbb{N}$ copies of B equipped with the norm $\|\cdot\|_k := \sup_{1\leq i\leq k} \|\cdot\|$. The product space B^k becomes again a Banach space. We introduce the following definition of k-linear maps, that will play an important role in the sequel:

Definition 8. *We say that $\mathcal{L}_k : B^k \to B$ for $k \in \mathbb{N}$ is a* `k-linear map` *if it is linear in each argument in the sense that for any $x_1, x_2, \ldots, x_k, y \in B$ and $a, b \in \mathbb{F}$*

$$\begin{aligned}\mathcal{L}_k(x_1, \ldots, x_{j-1}, &ax_j + by, x_{j+1}, \ldots, x_k)\\ &= a\mathcal{L}_k(x_1, \ldots, x_k) + b\mathcal{L}_k(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_k)\end{aligned}$$

for each $j = 1, \ldots, k$. A k-linear map $\mathcal{L}_k$ is `bounded` *if there exists a constant $K > 0$ such that*

$$\|\mathcal{L}_k(x_1, \ldots, x_k)\| \leq K\|x_1\| \cdots \|x_k\|$$

for all $x_1, \ldots, x_k \in B$. We denote the space of bounded k-linear maps by $L_k(B)$.

Notice that $L_1(B) = L(B)$, the space of bounded linear operators. Often we will call a k-linear map simply multilinear without specifying the dimension.

A k-linear map $\mathcal{L}_k$ induces a *k-monomial* $\mathcal{M}_k : B \to B$ by

$$\mathcal{M}_k(x) := \mathcal{L}_k(x, \ldots, x). \tag{5}$$

If $\mathcal{L}_k \in L_k(B)$, we see that $\|\mathcal{M}_k(x)\| \leq K\|x\|^k$, and we denote the set of all such k-monomials by $M_k(B)$. Of course, $M_1(B) = L(B)$, the space of bounded operators.

Additionally, we define $M_0(B) := B$ for completeness. $M_0(B)$ will play the role as the space of "constants", or, zero-order monomials. We remark that $M_k(B)$ is a vector space over the same field as B. We have the following result showing that the monomials are locally Lipschitz continuous on B:

Proposition 9. *If $\mathcal{M}_k \in M_k(B)$, then for any $x, y \in B$*

$$\|\mathcal{M}_k(x) - \mathcal{M}_k(y)\| \leq C(\|x\|, \|y\|)\|x - y\|$$

where $C(\|x\|, \|y\|) = K \sum_{i=1}^{k} \|x\|^{k-i}\|y\|^{i-1}$ for some positive constant K.

Proof. We notice that for $k = 1$, $\mathcal{M}_1 \in L(B)$ and therefore Lipschitz continuous. Let therefore $k \geq 2$. As $\mathcal{M}_k \in M_k(B)$, we have for $x \in B$ that $\mathcal{M}_k(x) = \mathcal{L}_k(x, \ldots, x)$ for a bounded k-linear map, $\mathcal{L}_k \in L_k(B)$. By adding and subtracting $\mathcal{L}_k(y, \ldots, y, x, \ldots, x)$, where $y \in B$ goes successively through all the $k - 1$ first coordinates, we find from the triangle inequality and the multilinearity property of $\mathcal{L}_k$,

$$\begin{aligned}
\|\mathcal{M}_k(x) - \mathcal{M}_k(y)\| &= \|\mathcal{L}_k(x, \ldots, x) - \mathcal{L}_k(y, \ldots, y)\| \\
&\leq \|\mathcal{L}_k(x, \ldots, x) - \mathcal{L}_k(y, x, \ldots, x)\| \\
&\quad + \|\mathcal{L}_k(y, x, \ldots, x) - \mathcal{L}_k(y, y, x, \ldots, x)\| \\
&\quad + \cdots \\
&\quad + \|\mathcal{L}_k(y, \ldots, y, x) - \mathcal{L}_k(y, \ldots, y)\| \\
&= \|\mathcal{L}_k(x - y, \ldots, x)\| \\
&\quad + \|\mathcal{L}_k(y, x - y, \ldots, x)\| \\
&\quad + \cdots \\
&\quad + \|\mathcal{L}_k(y, \ldots, y, x - y)\| \\
&\leq K\|x - y\|\|x\|^{k-1} + K\|y\|\|x - y\|\|x\|^{k-2} + \ldots \\
&\quad + K\|y\|^{k-1}\|x - y\|.
\end{aligned}$$

The last inequality follows from the boundedness of $\mathcal{L}_k$. The result follows. □

Let $(X(t))_{t\geq 0}$ be a B-valued stochastic process, that is, a family of B-valued random variables $X(t)$ indexed by $t \geq 0$. In the following we shall be interested in the conditional expectation $\mathcal{M}_k(X(t))$ given $\mathcal{F}_s$ for $t \geq s \geq 0$ where $\mathcal{M}_k \in M_k(B)$. More specifically, we want to define and study processes $(X(t))_{t\geq 0}$ where for any $\mathcal{M}_k \in M_k(B)$ there exists a family of jth-order monomials $\overline{\mathcal{M}}_j \in M_j(B)$ with $j \leq k$ such that

$$\mathbb{E}[\mathcal{M}_k(X(t)) \,|\, \mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j(X(s; t)), \tag{6}$$

and where $X(s; t)$ is some strongly $\mathcal{F}_s$-measurable random variable. As we see, we are interested in processes which preserve the "polynomial" order, as the monomials on the right hand side are not exceeding k in their orders. Moreover, the jth-order monomials $\overline{\mathcal{M}}_j$ are allowed to depend (deterministically) on s and t, however, we do not state this explicitly to lessen the notational burden.

A minimal requirement for studying (6) is that $\mathcal{M}_k(X(t))$ is Bochner integrable. As $X(t)$ is strongly measurable and $\mathcal{M}_k$ is continuous by Proposition 9, it follows from Lemma 2 that $\mathcal{M}_k(X(t))$ is strongly measurable. We introduce the following assumption:

Assumption 10. *The process $(X(t))_{t\geq 0}$ has finite moments of all order, i.e., for any $n \in \mathbb{N}$, $\mathbb{E}[\|X(t)\|^n] < \infty$ for all $t \geq 0$.*

Since we have $\|\mathcal{M}_k(x)\| \leq K\|x\|^k$, it follows under Assumption 10 that $\mathcal{M}_k(X(t))$ is Bochner integrable, and in particular the conditional expectation in (6) exists.

As a simple example, let us look at the case $B = \mathbb{R}$ and the function $\mathcal{M}_k : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{M}_k(x) = x^k$. Then one easily observes that $\mathcal{M}_k$ is induced by the k-linear map $\mathcal{L}_k : \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$, $(x_1, \ldots, x_k) \to x_1 \cdot \cdots \cdot x_k$. In Cuchiero et al. [9] and Filipović and Larsson [13], a real-valued $\mathcal{F}_t$-adapted stochastic process $(X(t))_{t\geq 0}$ is called a *polynomial process* if for any $n \in \mathbb{N}$, $\mathbb{E}[(X(t))^n \,|\, \mathcal{F}_s] = q_n(X(s))$ for some polynomial q_n of degree at most n. We will later see that k-linear maps arise naturally when dealing with polynomials in possibly non-commutative Banach algebras.

Next, let us define multilinear processes:

Definition 11. *Let $(X(t))_{t\geq 0}$ be a B-valued stochastic process and $(X(s;t))_{0\leq s\leq t<\infty}$ a family of B-valued random variables, such that $X(s;t)$ is $\mathcal{F}_s$-measurable. The process $(X(t))_{t\geq 0}$ is said to be a* `multilinear process` *with respect to the family $(X(s;t))_{0\leq s\leq t<\infty}$ if for any $k \in \mathbb{N}$ and $\mathcal{M}_k \in M_k(B)$, there exists a family of jth-order monomials $\overline{\mathcal{M}}_j \in M_j(B)$, $j \leq k$, such that for all $s \leq t$ it holds,*

$$\mathbb{E}[\mathcal{M}_k(X(t)) \,|\, \mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j(X(s;t)). \tag{7}$$

We remark that if we take a linear combination of monomials up to order $k \in \mathbb{N}$, that is, $\mathcal{P}_k := \sum_{j=0}^{k} p_j \mathcal{M}_j$ for $p_j \in \mathbb{F}$ and $\mathcal{M}_j \in M_j(B)$ for $j = 0, \ldots, k$, we find by the vector space structure of $M_j(B)$ that $\mathcal{P}_k$ can be represented by a sum of monomials up to degree k. Hence, we can use a linear combination of monomials in the conditional expectation defining a multilinear process in Definition 11. Further, we notice that we claim the existence of a familiy $(X(s;t))_{0\leq s\leq t<\infty}$ of $\mathcal{F}_s$-measurable random variables in the definition, rather than using $X(s)$ as argument on the right-hand side in (7).

An interesting class of processes is the *independent increment processes*, defined on general Banach spaces as:

Definition 12. *The process $(W(t))_{t\geq 0}$ is called an* `independent increment process` *if*

1. *$W(t)$ is strongly $\mathcal{F}_t$-measurable for any $t \geq 0$,*
2. *for every t and every $s \leq t$, there exists a decomposition of $W(t)$ into a strongly $\mathcal{F}_s$-measurable part $W^{\|}(s;t)$ and a part $W^{\perp}(s;t)$ that is independent of $\mathcal{F}_s$ such that $W(t) = W^{\perp}(s;t) + W^{\|}(s;t)$,*
3. *all moments of $\|W^{\perp}(s;t)\|$ and $\|W^{\|}(s;t)\|$ are integrable.*

Applebaum [3] defines a Lévy process on a separable Banach spaces as a B-valued stochastically continuous process $(L(t))_{t\geq 0}$ which is $\mathcal{F}_t$-adapted, the increments $L(t)-L(s)$ are independent of $\mathcal{F}_s$ for any $t > s \geq 0$ with distribution only depending on $t-s$, and having càdlàg paths. In view of Definition 12, $(L(t))_{t\geq 0}$ will be an independent increment process with $L^{\perp}(s;t) := L(t) - L(s)$ and $L^{\|}(s;t) := L(s)$

as long as all moments of $\|L(t)\|$ are integrable for any $t \geq 0$. Property (3) in Definition 12 follows by the fact that $L(t) - L(s) \stackrel{d}{=} L(t-s)$ by definition of the Lévy process. The canonical example of a Lévy process is the Wiener process. In a separable Banach space, Fernique's Theorem (see Peszat and Zabczyk [17]) also ensures the moment condition (3) in Definition 12 for a Wiener process. We provide several more examples of independent increment processes later. Also note that Property (3) in Definition 12 implies especially that $\mathbb{E}[\|W(t)\|^n] < \infty$ for all $n \in \mathbb{N}$.

Proposition 13. *Suppose that $(W(t))_{t\geq 0}$ is an independent increment process and let $\mathcal{M}_k \in M_k(B)$. Then there exists a family of jth-order monomials $\overline{\mathcal{M}}_j \in M_j(B)$, $0 \leq j \leq k$, such that*

$$\mathbb{E}[\mathcal{M}_k(W(t)) \,|\, \mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j(W^{\|}(s;t)), \tag{8}$$

for any $s \leq t$, where the $\overline{\mathcal{M}}_j$'s depend on s and t. In other words, $(W(t))_{t\geq 0}$ is a multilinear process with respect to $(W^{\|}(s;t))_{0\leq s\leq t<\infty}$.

Proof. For $k = 0$ the claim is trivial, so assume that $k \geq 1$. Let $\mathcal{L}_k \in L_k(B)$ be such that $\mathcal{M}_k(v) = \mathcal{L}_k(v, \dots, v)$. Recall by Definition 12 that $W(t) = W^{\perp}(s;t) + W^{\|}(s;t)$ for all $0 \leq s \leq t < \infty$, where $W^{\|}(s;t)$ is strongly $\mathcal{F}_s$-measurable and $W^{\perp}(s;t)$ is independent of $\mathcal{F}_s$. Thus we get from multilinearity of $\mathcal{L}_k$

$$\begin{aligned}
\mathcal{M}_k(W(t)) &= \mathcal{L}_k(W(t), \dots, W(t)) \\
&= \mathcal{L}_k(W^{\perp}(s;t) + W^{\|}(s;t), \dots, W^{\perp}(s;t) + W^{\|}(s;t)) \\
&= \mathcal{L}_k(W^{\|}(s;t), W^{\perp}(s;t) + W^{\|}(s;t), \dots, W^{\perp}(s;t) + W^{\|}(s;t)) \\
&\quad + \mathcal{L}_k(W^{\perp}(s;t), W^{\perp}(s;t) + W^{\|}(s;t), \dots, W^{\perp}(s;t) + W^{\|}(s;t)).
\end{aligned}$$

Continuing like this over the remaining $k-1$ arguments one can decompose the above expression into a linear combination of 2^k terms of the form

$$\mathcal{L}_k(X_{j,1}, \dots, X_{j,k})$$

for $1 \leq j \leq 2^k$ with $X_{j,i} \in \{W^{\perp}(s;t), W^{\|}(s;t)\}$, and there are exactly $\binom{k}{n}$ terms j for which $\#\{X_{j,i} \,|\, X_{j,i} = W^{\|}(s;t)\} = n$.

Let us look at a particular term where $W^{\|}(s;t)$ appears in the first two arguments. Introduce the function $\mathcal{L}_{2,1} : B \times B \to B$ defined as

$$\mathcal{L}_{2,1}(y_1, y_2) = \mathbb{E}[\mathcal{L}_k(y_1, y_2, W^{\perp}(s;t), \dots, W^{\perp}(s;t))].$$

The subscript $(2,1)$ denotes that $\mathcal{L}_{2,1}$ is the function related to the first term in which $W^{\|}(s;t)$ appears twice, where the ordering is irrelevant. In view of Proposition 4, let $f(x,y) = \mathcal{L}_k(y, y, x, \dots, x)$, $X = W^{\perp}(s;t)$ and $Y = W^{\|}(s;t)$. Then, $\sigma(X) = \sigma(W^{\perp}(s;t))$ and $\mathcal{F}_s$ are independent, and $Y = W^{\|}(s;t)$ is strongly $\mathcal{F}_s$-measurable. First, we show that $(x,y) \mapsto f(x,y)$ is continuous: Indeed, for

$(x,y),(u,v)\in B\times B$, we find by triangle inequality and $\mathcal{L}_k\in L_k(B)$ that

$$\begin{aligned}
\|f(u,v)-f(x,y)\| &= \|\mathcal{L}_k(v,v,u,\dots,u)-\mathcal{L}_k(y,y,x,\dots,x)\| \\
&\le \|\mathcal{L}_k(v,v,u,\dots,u)-\mathcal{L}_k(y,v,u,\dots,u)\| \\
&\quad + \|\mathcal{L}_k(y,v,u,\dots,u)-\mathcal{L}_k(y,y,u,\dots,u)\| \\
&\quad + \|\mathcal{L}_k(y,y,u,\dots,u)-\mathcal{L}_k(y,y,x,u,\dots,u)\| \\
&\quad \cdots \\
&\quad + \|\mathcal{L}_k(y,y,x,\dots,x,u)-\mathcal{L}_k(y,y,x,\dots,x)\| \\
&= \|\mathcal{L}_k(v-y,v,u,\dots,u)\| + \|\mathcal{L}_k(y,v-y,u,\dots,u)\| \\
&\quad + \|\mathcal{L}_k(y,y,u-x,u,\dots,u)\| \\
&\quad + \cdots \\
&\quad + \|\mathcal{L}_k(y,y,x,\dots,x,u-x)\| \\
&\le K\|v-y\|\|v\|\|u\|^{k-2} + K\|y\|\|v-y\|\|u\|^{k-2} \\
&\quad + K\|y\|^2\|u-x\|\|u\|^{k-3} + \cdots + K\|y\|^2\|u-x\|\|x\|^{k-2} \\
&= K\|u\|^{k-2}(\|v\|+\|y\|)\|v-y\| \\
&\quad + K\|y\|^2\Big(\sum_{n=0}^{k-3}\|x\|^n\|u\|^{k-3-n}\Big)\|u-x\|
\end{aligned}$$

Thus, $(x,y)\mapsto f(x,y)$ is a local Lipschitz continuous map from $B\times B$ into B.

Continuity implies that $f(X,y)$ is strongly measurable (see Lemma 2). Since

$$\|f(X,y)\| \le K\|y\|^2\|X\|^{n-2} \tag{9}$$

it follows that $f(X,y)$ is Bochner integrable by the finite moments condition on $\|X\|=\|W^{\perp}(s;t)\|$. Furthermore,

$$\|f(X,\tilde{Y})]\| \le K\|\tilde{Y}\|^2\|X\|^{k-2} \le K\|Y\|^2\|X\|^{k-2} =: \tilde{Z} \tag{10}$$

provided that $\|\tilde{Y}\|\le\|Y\|$ and therefore the bound in (3) holds. Again using (9) it follows that $\|\mathbb{E}[f(X,y)]\|\le K\|y\|^2\mathbb{E}[\|X\|^{k-2}]$ and

$$\|\mathbb{E}[f(X,y)]_{y=\tilde{Y}}\| \le K\|\tilde{Y}\|^2\mathbb{E}[\|X\|^{k-2}] \le K\|Y\|^2\mathbb{E}[\|X\|^{k-2}] =: Z$$

provided that $\|\tilde{Y}\|\le\|Y\|$ and the bound in (2) holds. Moreover, appealing to the fact that $\mathcal{L}_k\in L_k(B)$ and Bochner's inequality together with the finiteness of all moments of $\|X\|$, we find by using similar arguments as above that $y\mapsto\mathbb{E}[f(X,y)]$ is locally Lipschitz continuous. Thus we can apply Proposition 4 and conclude that

$$\begin{aligned}
\mathcal{L}_{2,1}(W^{\|}(s;t),W^{\|}(s;t)) &= \mathbb{E}[\mathcal{L}_k(y,y,W^{\perp}(s;t),\dots,W^{\perp}(s;t))]_{y=W^{\|}(s;t)} \\
&= \mathbb{E}[\mathcal{L}_k(W^{\|}(s;t),W^{\|}(s;t),W^{\perp}(s;t),\dots,W^{\perp}(s;t))\,|\,\mathcal{F}_s].
\end{aligned}$$

By linearity of the expectation operator along with multilinearity of $\mathcal{L}_k$, the function $\mathcal{L}_{2,1}(y_1,y_2)$ is bilinear and indeed an element of $L_2(B)$. The same argument applies to the other $\binom{k}{2}-1$ terms $\mathcal{L}_{2,2},\dots,\mathcal{L}_{2,\binom{k}{2}}$ with $W^{\|}(s;t)$ appearing twice. Since the sum of bilinear maps is bilinear we can define the bilinear function:

$$\widetilde{\mathcal{L}}_2(y_1,y_2) = \sum_{i=1}^{\binom{k}{2}}\mathcal{L}_{2,i}(y_1,y_2)$$

and $\overline{\mathcal{M}}_2(y) := \widetilde{\mathcal{L}}_2(y, y) \in M_2(B)$. In the same way the other functions $\overline{\mathcal{A}}_j(W^{\|}(s;t))$ can be defined for $j \in \{1, 3, \ldots, k\}$ and the representation (8) follows. Thus, the proposition is proved. □

We next observe that elements of $M_k(B)$ share similar characteristics as the monomials on the real line. In fact their $k+1$-th Fréchet derivative vanishes.

Proposition 14. *Assume for $k \in \mathbb{N}$ that $\mathcal{M}_k \in M_k(B)$ is induced by $\mathcal{L}_k \in L_k(B)$. Then the n-th Fréchet derivative $D^n\mathcal{M}_k : B \to L_n(B)$ is given by*

$$D^n\mathcal{M}_k(u)(h_1, \ldots, h_n) = \sum_{\substack{x_i \in \{u, h_1, \ldots, h_n\} \\ \#\{i \,|\, x_i = h_j\} = 1 \\ 1 \le j \le n}} \mathcal{L}_k(x_1, \ldots, x_k) \tag{11}$$

for $n = 1, \ldots, k$ and $D^{k+1}\mathcal{M}_k(u)(h_1, \ldots, h_{k+1}) = 0$.

Proof. We have $\mathcal{M}_k(u) = \mathcal{L}_k(u, \ldots, u)$. Using the chain rule for the Fréchet derivative we then get that

$$D\mathcal{M}_k(u) = \sum_{i=1}^{k} \nabla_i \mathcal{L}_k(u, \ldots, u) \cdot 1.$$

To calculate $\nabla_i \mathcal{L}_k(u, \ldots, u)$, observe that since $\mathcal{L}_k(u + h_1, \ldots, u) - \mathcal{L}_k(u, \ldots, u) - \mathcal{L}_k(h_1, \ldots, u) = 0$ by multilinearity and therefore by the definition of the Fréchet derivative

$$\lim_{\|h_1\| \to 0} \frac{\|\mathcal{L}_k(u + h_1, u, \ldots, u) - \mathcal{L}_k(u, u, \ldots, u) - \mathcal{L}_k(h_1, u, \ldots, u)\|}{\|h_1\|} = 0. \tag{12}$$

Hence, $\nabla_1 \mathcal{L}_k(u, \ldots, u)(h_1) = \mathcal{L}_k(h_1, u, \ldots, u)$, and more generally $\nabla_i \mathcal{L}_k(u, \ldots, u)(h_1) = \mathcal{L}_k(u, \ldots, u, h_1, u, \ldots, u)$, where the entry h_1 is in the i-th coordinate. It follows that,

$$D\mathcal{M}_k(u)(h_1) = \sum_{\substack{x_i \in \{u, h_1\} \\ \#\{i \,|\, x_i = h_1\} = 1}} \mathcal{L}_k(x_1, \ldots, x_k).$$

Clearly $D\mathcal{M}_k$ maps from B to $L_1(B) = L(B)$.

The claim now follows by induction: assume that (11) holds for $n < k$. We pick the term $\mathcal{L}_k(u, \ldots, u, h_1, \ldots, h_n)$ from the sum in (11). Then

$$\begin{aligned} D\mathcal{L}_k(u, \ldots, u, h_1, \ldots, h_n)(h_{n+1}) &= \sum_{i=1}^{k-n} \nabla_i \mathcal{L}_k(u, \ldots, u, h_1, \ldots, h_n)(h_{n+1}) \\ &= \mathcal{L}_k(h_{n+1}, u, \ldots, u, h_1, \ldots, h_n) \\ &\quad + \mathcal{L}_k(u, h_{n+1}, u, \ldots, u, h_1, \ldots, h_n) \\ &\quad + \cdots \\ &\quad + \mathcal{L}_k(u, \ldots, u, h_{n+1}, h_1, \ldots, h_n) \end{aligned}$$

and similarly with all the other terms in (11). We can then compute $DD^n\mathcal{M}_k = D^{n+1}\mathcal{L}_k$, from which the representation (11) follows for $n+1$. Directly from this representation one observes that $D^{n+1}\mathcal{M}_k$ can be seen as a map from B to $L_{n+1}(B)$. Finally, $D^k\mathcal{M}_k(u)$ is constant and therefore $D^{k+1}\mathcal{M}_k(u) = 0$. □

In view of this result, it is fair to call the elements in $M_k(B)$ *monomials*, as we have done.

Corollary 15. *Assume for $k \in \mathbb{N}$ that $\mathcal{M}_k \in M_k(B)$ is induced by $\mathcal{L}_k \in L_k(B)$. Then the n-th Fréchet derivative $D^n\mathcal{M}_k : B \to L_n(B)$ is symmetric for any $2 \leq n \leq k$, i.e.*

$$D^n\mathcal{M}_k(u)(h_1, \dots, h_n) = D^n\mathcal{M}_k(u)(h_{\sigma(1)}, \dots, h_{\sigma(n)}) \tag{13}$$

for any permutation $\sigma \in S_n$, where S_n denotes the set of permutations on $\{1, \dots, n\}$.

Proof. We can rewrite (11) as a double sum where we first fix the appearance of the u and then sum over those terms with u in the same coordinate. Again for notational simplicity we look at the specific one with the u fixed to be in the first $k - n$ coordinates and we find that

$$\begin{aligned}\sum_{\substack{i\in\{h_1,\dots,h_n\}\\ \#\{i\,|\,x_i=h_j\}=1\\ 1\leq j\leq n}} &\mathcal{L}_k(u, \dots, u, x_1, \dots, x_n)\\ &= \sum_{\sigma\in S_n} \mathcal{L}_k(u, \dots, u, h_{\sigma(1)}, \dots, h_{\sigma(n)})\end{aligned}$$

and therefore the expression is symmetric. The same argument works for any fixed positions for the u's and there are $\binom{k}{n}$ possible ways to fix them. Therefore $D^n\mathcal{M}_k(u)(h_1, \dots, h_n)$ is the sum of $\binom{k}{n}$ symmetric functions and is therefore itself symmetric. □

We immediately get the following Corollary which will be important later for polynomials in Banach algebras.

Corollary 16. *Let B be a Banach algebra and $\mathcal{L} \in L(B)$. Define the k-th order monomial $\mathcal{M}_k(u) = \mathcal{L}(u^k)$. For $n \leq k$, the n-th order Fréchet derivative $D^n\mathcal{M}_k : B \to L_n(B)$ of $\mathcal{M}_k$ is given by*

$$D^n\mathcal{M}_k(u)(h_1, \dots, h_n) = \mathcal{L}\left(\sum_{\substack{x_i\in\{u,h_1,\dots,h_n\}\\ \#\{i\,|\,x_i=h_j\}=1\\ 1\leq j\leq n}} x_1 \cdot \dots \cdot x_k\right). \tag{14}$$

Furthermore if B is commutative, then the expression simplifies to

$$D^n\mathcal{M}_k(u)(h_1, \dots, h_n) = \frac{k!}{(k-n)!}\mathcal{L}\left(h_1 \cdots h_n u^{k-n}\right). \tag{15}$$

Proof. The monomial $\mathcal{M}_k$ is induced from the multilinear map $\mathcal{L}_k : (u_1, \dots, u_k) \to \mathcal{L}(u_1 \cdot \dots \cdot u_k)$. Then (14) directly follows form Proposition 14. If B is commutative, then all terms appearing in the sum in (14) are equal. In fact there are $\binom{k}{n}$ ways to fix the appearance of the u and then $n!$ ways to distribute the $h_1, \dots, h_n$ in the remaining positions. So altogether there are $\frac{k!}{(k-n)!}$ equal terms and (15) follows. □

3.1. **Multilinear forms.** In this section we shall elaborate a bit on multilinear forms which map into the field $\mathbb{F}$ instead of the Banach space B. We first give a precise definition:

Definition 17. *We say that $\mathcal{L}_k : B^k \to \mathbb{F}$ for $k \in \mathbb{N}$ is a* `k-linear form` *if it is linear in each argument in the sense that for any $x_1, x_2, \dots, x_k, y \in B$ and $a, b \in \mathbb{F}$*

$$\begin{aligned}\mathcal{L}_k(x_1, \dots, x_{j-1}, &ax_j + by, x_{j+1}, \dots, x_k)\\ &= a\mathcal{L}_k(x_1, \dots, x_k) + b\mathcal{L}_k(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_k)\end{aligned}$$

for each $j = 1, \dots, k$. A k-linear form $\mathcal{L}_k$ is `bounded` *if there exists a constant $K > 0$ such that*

$$|\mathcal{L}_k(x_1, \dots, x_k)| \le K\|x_1\| \cdots \|x_k\|$$

for all $x_1, \dots, x_k \in B$. We denote the space of bounded k-linear forms by $L_k^{\mathbb{F}}(B)$.

Notice that $L_1^{\mathbb{F}}(B)$ is the dual space of B. A k-linear form $\mathcal{L}_k$ induces a *k-monomial* $\mathcal{M}_k : B \to \mathbb{F}$ by

$$\mathcal{M}_k(x) := \mathcal{L}_k(x, \dots, x). \tag{16}$$

If $\mathcal{L}_k \in L_k^{\mathbb{F}}(B)$, we see that $|\mathcal{M}_k(x)| \le K\|x\|^k$, and we denote the set of all such k-monomials by $M_k^{\mathbb{F}}(B)$. We use the convention that $M_0^{\mathbb{F}}(B) = \mathbb{F}$. Observe that $M_1^{\mathbb{F}}(B) = L_1^{\mathbb{F}}(B)$.

We show that the multilinearity preserving property implies that also monomials arising from multilinear forms are preserved. We use this result in Section 5 for the calculation of conditional moments for Hilbert space valued stochastic processes but the result might be of independent interest.

Proposition 18. *Let B be a Hilbert space with with inner product $\langle \cdot, \cdot \rangle$. Let $(X(t))_{t \ge 0}$ be a multilinear B-valued process with respect to the family of B-valued random variables $(X(s;t))_{0 \le s \le t < \infty}$. For every k monomial $\mathcal{M}_k \in M_k^{\mathbb{F}}(B)$, there exist j-monomials $\overline{\mathcal{M}}_j \in M_j^{\mathbb{F}}(B), j = 0, \dots, k$ such that*

$$\mathbb{E}[\mathcal{M}_k(X(t)) \,|\, \mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j(X(s;t)).$$

Proof. Let $\mathcal{M}_k \in M_k^{\mathbb{F}}(B)$. Choose $z \in B$ with $\|z\|_B = 1$. Define the k-monomial $\mathcal{M}_k^z \in M_k(B)$ by $\mathcal{M}_k^z(x) := z\mathcal{M}_k(x)$. Then there exist $\overline{\mathcal{M}}_j^z \in M_j(B)$ for $j = 0, \dots, k$ such that

$$\mathbb{E}[\mathcal{M}_k^z(X(t))|\mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j^z(X(s;t)).$$

Clearly, $\overline{\mathcal{M}}_j := \langle \overline{\mathcal{M}}_j^z(x), z \rangle$ defines an element in $M_j^{\mathbb{F}}(B)$. Now observe that

$$\begin{aligned}\mathbb{E}[\mathcal{M}_k(X(t))|\mathcal{F}_s] &= \langle \mathbb{E}[\mathcal{M}_k^z(X(t))|\mathcal{F}_s], z \rangle\\ &= \langle \sum_{j=0}^{k} \overline{\mathcal{M}}_j^z(X(s;t)), z \rangle\\ &= \sum_{j=0}^{k} \overline{\mathcal{M}}_j(X(s;t)).\end{aligned}$$

□

4. Multiplicative maps and polynomials

We shall now focus on Banach spaces B which are Banach algebras. We recall that when B is a Banach algebra, there is a multiplication operator $\cdot : B \times B \to B$ defined such that $(B, +, \cdot)$ is an associative $\mathbb{F}$-algebra and $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ for any $x, y \in B$.

Suppose that $(W(t))_{t\geq 0}$ is an independent increment process in B (see Definition 12), and recall the decomposition $W(t) = W_s^{\perp}(t) + W_s^{\|}(t)$ for any $0 \leq s \leq t$. According to Proposition 13, $(W(t))_{t\geq 0}$ is a multilinear process with respect to $(W^{\|}(s;t))_{0\leq s\leq t<\infty}$, i.e., for every $\mathcal{M}_k \in M_k(B), k \in \mathbb{N}$, there exist $\overline{\mathcal{M}}_j \in M_j(B), j = 0, \dots, k$ such that

$$\mathbb{E}[\mathcal{M}_k(W(t)) \,|\, \mathcal{F}_s] = \sum_{j=0}^{k} \overline{\mathcal{M}}_j(W_s^{\|}(t)), \tag{17}$$

for all $0 \leq s \leq t$. We now look at the particular case of $\mathcal{M}_k(x) = x^k$ and address the question under which conditions the induced j-order monomials $\overline{\mathcal{M}}_j$ in (17) are of polynomial type as well.

Lemma 19. *Assume that B is a commutative Banach algebra. Then for all $0 \leq s \leq t$*

$$\mathbb{E}[W^k(t) \,|\, \mathcal{F}_s] = q_k(W^{\|}(s;t))$$

with

$$q_k(x) = \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}[(W^{\perp}(s;t))^{k-n}] x^n.$$

Here, we use the convention that $x^0 = 1 \in \mathbb{F}$, that is, the term $b_0 x^0 = b_0 \in B$.

Proof. Let $k \in \mathbb{N}$. By the binomial formula, we find for $s \leq t$

$$W^k(t) = (W^{\perp}(s;t) + W^{\|}(s;t))^k = \sum_{n=0}^{k} \binom{k}{n} (W^{\perp}(s;t))^{k-n} (W^{\|}(s;t))^n.$$

We get by $\mathcal{F}_t$-adaptedness of the process and Lemma 6 that $(W^{\|}(s;t))^n$ is strongly $\mathcal{F}_s$-measurable for all $n \leq k$. From Proposition 7 above,

$$\begin{aligned}
\mathbb{E}[W^k(t) \,|\, \mathcal{F}_s] &= \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}\left[(W^{\perp}(s;t))^{k-n} (W^{\|}(s;t))^n \,|\, \mathcal{F}_s\right] \\
&= \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}\left[(W^{\perp}(s;t))^{k-n} \,|\, \mathcal{F}_s\right] (W^{\|}(s;t))^n \\
&= \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}\left[(W^{\perp}(s;t))^{k-n}\right] (W^{\|}(s;t))^n.
\end{aligned}$$

In the last step, we used independence of the increments and Lemma 3. Thus, the lemma follows. □

If B is commutative, define a polynomial $p_k : B \to B$ of order $k \in \mathbb{N}$, as

$$p_k(x) = \sum_{n=0}^{k} b_n x^n \,, \tag{18}$$

where $(b_n)_{n=0}^k \subset B$ and with the convention that $x^0 = 1 \in \mathbb{F}$, i.e., $b_0 x^0 = b_0 \in B$. If $x \in B$, we find

$$\|p_k(x)\| \leq \sum_{n=0}^k \|b_n\| \|x\|^n < \infty$$

by the triangle inequality and Banach algebra norm. We denote the space of polynomials in B of order k by $\mathrm{Pol}_k(B)$. If $B = \mathbb{R}$, $\mathrm{Pol}_k(\mathbb{R})$ is the space of polynomials on the real line of order k.

From Lemma 19 it is simple to see that in a commutative Banach algebra

$$\mathbb{E}[p_k(W(t)) \,|\, \mathcal{F}_s] = \sum_{n=0}^k b_n q_n(W^{\|}(s;t)) = \widetilde{q}_k(W^{\|}(s;t))$$

for any $k \in \mathbb{N}$ and $0 \leq s \leq t$, where $\widetilde{q}_k \in \mathrm{Pol}_k(B)$ is given by

$$\widetilde{q}_k(x) = \sum_{n=0}^k b_n \sum_{j=0}^n \binom{n}{j} \mathbb{E}[(W^{\perp}(s;t))^{n-j}] x^j.$$

This motivates a definition of a *polynomial process* in a commutative Banach algebra B:

Definition 20. *Let $(X(t))_{t\geq 0}$ be a B-valued stochastic process where B is a commutative Banach algebra. Furthermore, let $(X(s;t))_{0\leq s\leq t<\infty}$ be a family of B-valued random variables, such that $X(s;t)$ is strongly $\mathcal{F}_s$-measurable for all $0 \leq s \leq t$. The process $(X(t))_{t\geq 0}$ is said to be a* `polynomial process` *with respect to the family $(X(s;t))_{0\leq s\leq t<\infty}$ if for all $s \leq t$, $k \in \mathbb{N}$ and every polynomial $p_k \in Pol_k(B)$ there exists a polynomial $q_m \in Pol_m(B)$, $m \leq k$ such that*

$$\mathbb{E}[p_k(X(t)) \,|\, \mathcal{F}_s] = q_m(X(s;t)).$$

Note that the coefficients of q_m may depend on the times s and t. We summarize our findings from above in the following Proposition.

Proposition 21. *Assume that B is a commutative Banach algebra. Then the independent increment process $(W(t))_{t\geq 0}$ defined in Definition 12 is a polynomial process with respect to $(W^{\|}(s;t))_{0\leq s\leq t<\infty}$.*

Observe that any $b \in B$ gives rise to a multiplication operator $B \ni x \mapsto bx \in B$, being a bounded linear operator. Hence, the above definition of polynomials may be viewed as a special case of more general polynomials with $b_n \in L(B)$, as we define next. Define a *generalized* polynomial $\mathcal{P}_k : B \to B$ of order $k \in \mathbb{N}$, as

$$\mathcal{P}_k(x) = \sum_{n=0}^k \mathcal{B}_n(x^n), \tag{19}$$

where $(\mathcal{B}_n)_{n=1}^k \subset L(B)$ and $\mathcal{B}_0 \in B$ is a constant reflecting the fact that $x^0 = 1 \in \mathbb{F}$. If $x \in B$, we find

$$\|\mathcal{P}_k(x)\| \leq \sum_{n=0}^k \|\mathcal{B}_n\|_{\mathrm{op}} \|x\|^n < \infty$$

by the triangle inequality and Banach algebra norm. Here, $\|\mathcal{B}_n\|_{\mathrm{op}}$ denotes the operator norm of $\mathcal{B}_n$. We denote the space of generalized polynomials in B of order k by $\mathrm{gPol}_k(B)$. If $B = \mathbb{R}$, $\mathrm{gPol}_k(\mathbb{R}) = \mathrm{Pol}_k(\mathbb{R})$ is the space of polynomials on the real line of order k.

We define the following

Definition 22. *Let $(X(t))_{t\geq 0}$ be a B-valued stochastic process where B is a Banach algebra. Furthermore, let $(X(s;t))_{0\leq s\leq t<\infty}$ be a family of B-valued random variables, such that $X(s;t)$ is strongly $\mathcal{F}_s$-measurable for all $0 \leq s \leq t$. The process $(X(t))_{t\geq 0}$ is said to be a* `generalized polynomial process` *with respect to the family $(X(s;t))_{0\leq s\leq t<\infty}$ if for all $s \leq t$, $k \in \mathbb{N}$ and every generalized polynomial $\mathcal{P}_k \in gPol_k(B)$ there exists a generalized polynomial $\mathcal{Q}_m \in gPol_m(B)$, $m \leq k$ such that*

$$\mathbb{E}[\mathcal{P}_k(X(t))\,|\,\mathcal{F}_s] = \mathcal{Q}_m(X(s;t)).$$

Remark that in the above definition, we do not assume that B is commutative.

The name *generalized polynomial process* is justified by the following Proposition which states that every polynomial process is also a generalized polynomial process.

Proposition 23. *Assume B is a commutative Banach algebra, and $(X(t))_{t\geq 0}$ is a polynomial process in B with respect to the family $(X(s;t))_{0\leq s\leq t<\infty}$. Then $(X(t))_{t\geq 0}$ is also a generalized polynomial process in B with respect to the family $(X(s;t))_{0\leq s\leq t<\infty}$.*

Proof. For $k \in \mathbb{N}$, let $\mathcal{P}_k \in \mathrm{gPol}_m(B)$ with

$$\mathcal{P}_k(x) = \sum_{n=0}^{k} \mathcal{B}_n(x^n)\,. \tag{20}$$

Because $(X(t))_{t\geq 0}$ is a polynomial process with respect to $(X(s;t))_{0\leq s\leq t<\infty}$, it follows that for each $0 \leq n \leq k$ there exists a $q_{m(n)} \in \mathrm{Pol}_{m(n)}(B)$ for $m(n) \leq k$ such that for any $s \leq t$

$$\mathbb{E}[(X(t))^n\,|\,\mathcal{F}_s] = q_{m(n)}(X(s;t)) = \sum_{j=0}^{m(n)} b_{n,j}(X(s;t))^j \tag{21}$$

where $(b_{n,j})_{j=0,\ldots,m(n)} \subset B$. It follows using Lemma 1 that

$$\begin{aligned}\mathbb{E}[\mathcal{B}_n((X(t))^n)\,|\,\mathcal{F}_s] &= \mathcal{B}_n(\mathbb{E}[((X(t))^n)\,|\,\mathcal{F}_s]) \\ &= \mathcal{B}_n(\sum_{j=0}^{m(n)} b_{n,j}(X(s;t))^j) \\ &= \sum_{j=0}^{m(n)} \mathcal{B}_n(b_{n,j}(X(s;t))^j) \\ &= \sum_{j=0}^{m(n)} \widetilde{\mathcal{B}}_{n,j}(X(s;t))^j\end{aligned}$$

with $\widetilde{\mathcal{B}}_{n,j} \in L(B)$ being defined by $B \ni x \mapsto \mathcal{B}_n(b_{n,j}x) \in B$. Define the generalized polynomial $\mathcal{Q}_{\widehat{m}} \in \mathrm{gPol}_{\widehat{m}}(B)$ by $\mathcal{Q}_{\widehat{m}}(x) = \sum_{n=0}^{k}\sum_{j=0}^{m(n)} \widetilde{\mathcal{B}}_{n,j}x^j$ with $\widehat{m} := \max_{0\leq n\leq k} m(n) \leq k$. The result follows. $\square$

To see that the opposite does not hold in general we shall look at an example, which is interesting from the application point of view. To this end, assume that the commutative Banach algebra B is a separable Hilbert space. Let $(\mathcal{S}_t)_{t\geq 0}$ be a C_0-semigroup on B and $(W(t))_{t\geq 0}$ a B-valued Wiener process. By Fernique's Theorem all moments of $\|W(t)\|$ are finite. We recall that $(W(t))_{t\geq 0}$ is an independent increment process by Definition 12. Consider the stochastic process $(X(t))_{t\geq 0}$ given by

$$X(t) = \int_0^t \mathcal{S}_{t-s}\, dW(s). \tag{22}$$

As $(W(t))_{t\geq 0}$ in particular is square-integrable, it follows that the stochastic convolution $(X(t))_{t\geq 0}$ is a well-defined $\mathcal{F}_t$-adapted process in B (see Applebaum [3] and Peszat and Zabczyk [17]). Moreover, it is known (see again Applebaum [3] and Peszat and Zabczyk [17]) that $(X(t))_{t\geq 0}$ is a mild solution of the stochastic evolution equation

$$dX(t) = \mathcal{A}X(t)\, dt + dW(t) \tag{23}$$

where $\mathcal{A}$ is the (densely defined) generator of $(\mathcal{S}_t)_{t\geq 0}$.

We decompose $X(t)$ in (22) into $X^{\perp}(s;t) := \int_s^t \mathcal{S}_{t-u} dW(u)$ and $X^{\|}(s;t) := \int_0^s \mathcal{S}_{t-u} dW(u)$. We find that $X^{\perp}(s;t)$ is independent of $\mathcal{F}_s$ and $X^{\|}(s;t)$ is $\mathcal{F}_s$-measurable. Hence, $(X(t))_{t\geq 0}$ is an independent increment process in B. Then, by Proposition 21, it holds for any $k \in \mathbb{N}$ and $s \leq t$

$$\mathbb{E}[X^k(t) \,|\, \mathcal{F}_s] = \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}[(X^{\perp}(s;t))^{k-n}](X^{\|}(s;t))^n.$$

Hence, as expected, $(X(t))_{t\geq 0}$ is a polynomial process with respect to the family $(X^{\|}(s;t))_{0\leq s\leq t<\infty}$. Let us analyse the situation a few steps further: From the semigroup property of $(\mathcal{S}_t)_{t\geq 0}$, we find that

$$X^{\|}(s;t) = \mathcal{S}_{t-s} \int_0^s \mathcal{S}_{s-u} dW(u) = \mathcal{S}_{t-s} X(s).$$

Thus,

$$\mathbb{E}[X^k(t) \,|\, \mathcal{F}_s] = \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}[(X^{\perp}(s;t))^{k-n}](\mathcal{S}_{t-s}X(s))^n.$$

Now, assume $(\mathcal{S}_t)_{t\geq 0}$ is a homomorphism of algebras so that $\mathcal{S}_t(x\cdot y) = (\mathcal{S}_t x)\cdot(\mathcal{S}_t y)$ for any $x, y \in B$. Then $(\mathcal{S}_t x)^n = \mathcal{S}_t x^n$ for all $n \in \mathbb{N}$ and we find

$$\mathbb{E}[X^k(t) \,|\, \mathcal{F}_s] = \sum_{n=0}^{k} \binom{k}{n} \mathbb{E}[(X^{\perp}(s;t))^{k-n}]\mathcal{S}_{t-s} X^n(s). \tag{24}$$

This shows that $(X(t))_{t\geq 0}$ is a *generalized* polynomial process with respect to the family $(X(s))_{0\leq s\leq t<\infty}$. This is in line with the definition of finite-dimensional polynomial processes (see Cuchiero et al. [9] and Filipović and Larsson [13]), and the fact that we can establish the generalized polynomial preserving property of the process with respect to itself is significantly stronger and more applicable than merely in terms of some family of $\mathcal{F}_s$-measurable random variables. On the other hand, $(X(t))_{t\geq 0}$ is *not* a polynomial process with respect to $(X(s))_{0\leq s\leq t<\infty}$, as the coefficients on the right hand side of (24) are elements in $L(B)$. This provides us with an example of a process which is generalized polynomial but not polynomial.

It is worth emphasising that the above analysis shows that in general stochastic convolutions as in (22) are polynomial processes with respect to $(\mathcal{S}_{t-s}X(s))_{0\leq s\leq t<\infty}$. Indeed, they are generalized polynomial processes with respect to $(X(s))_{0\leq s\leq t<\infty}$ when the semigroup is a homomorphism, but not polynomial with respect to the same family. Hence, the extension of Ornstein-Uhlenbeck processes to infinite dimensions as in (23) is not straightforwardly preserving the natural polynomial property from the finite-dimensional case.

The class of processes defined in (22) is of interest from the application point of view. Stochastic evolution equation like the Ornstein-Uhlenbeck process in (23) appear in many applications, for example as the heat equation in random media (see e.g. Walsh [18]) or as the dynamics of forward prices in finance and commodity markets (see e.g. Benth and Krühner [4]). We return to the latter in the next Section.

4.1. **Counterexample: non-commutative case.** Consider the case when B is a non-commutative Banach algebra. Then the binomial formula used in the proof of the Lemma 19 and later above does not hold. For example, if $(W(t))_{t\geq 0}$ is an independent increment process in B, we find for $t \geq s$ that

$$\begin{aligned}
\mathbb{E}[W^3(t)\,|\,\mathcal{F}_s] &= \mathbb{E}[(W^{\perp}(s;t))^3\,|\,\mathcal{F}_s] + \mathbb{E}[(W^{\perp}(s;t))^2 W^{\|}(s;t)\,|\,\mathcal{F}_s] \\
&\quad + \mathbb{E}[(W^{\perp}(s;t))W^{\|}(s;t)(W^{\perp}(s;t))\,|\,\mathcal{F}_s] + \mathbb{E}[(W^{\perp}(s;t))(W^{\|}(s;t))^2\,|\,\mathcal{F}_s] \\
&\quad + \mathbb{E}[W^{\|}(s;t)(W^{\perp}(s;t))^2\,|\,\mathcal{F}_s] + \mathbb{E}[W^{\|}(s;t)(W^{\perp}(s;t))W^{\|}(s;t)\,|\,\mathcal{F}_s] \\
&\quad + \mathbb{E}[(W^{\|}(s;t))^2(W^{\perp}(s;t))\,|\,\mathcal{F}_s] + \mathbb{E}[(W^{\|}(s;t))^3\,|\,\mathcal{F}_s] \\
&= \mathbb{E}[(W^{\perp}(s;t))^3] + E[(W^{\perp}(s;t))^2]W^{\|}(s;t) \\
&\quad + \mathbb{E}[(W^{\perp}(s;t))W^{\|}(s;t)(W^{\perp}(s;t))\,|\,\mathcal{F}_s] \\
&\quad + \mathbb{E}[(W^{\perp}(s;t))](W^{\|}(s;t))^2 + W^{\|}(s;t)\mathbb{E}[(W^{\perp}(s;t))^2] \\
&\quad + (W^{\|}(s;t))^2\mathbb{E}[(W^{\perp}(s;t))] + (W^{\|}(s;t))^3
\end{aligned}$$

after appealing to independence and measurability using Lemmas 3 and 7. It is not immediately clear how to deal with the term involving the conditional expectation of $W^{\perp}(s;t)W^{\|}(s;t)W^{\perp}(s;t)$, and thus how to express $\mathbb{E}[W^3(t)\,|\,\mathcal{F}_s]$ as a polynomial in $W^{\|}(s;t)$.

Using Proposition 4, we know that

$$\mathbb{E}[(W^{\perp}(s;t))W^{\|}(s;t)(W^{\perp}(s;t))\,|\,\mathcal{F}_s] = \mathbb{E}[(W^{\perp}(s;t))y(W^{\perp}(s;t))]_{y=W^{\|}(s;t)}$$

and observe that $B \ni y \mapsto \mathbb{E}[(W^{\perp}(s;t))y(W^{\perp}(s;t))] \in B$ is a linear function. Furthermore, it is bounded as

$$\begin{aligned}
\|\mathbb{E}[(W^{\perp}(s;t))y(W^{\perp}(s;t))]\| &\leq \mathbb{E}[\|(W^{\perp}(s;t))y(W^{\perp}(s;t))\|] \\
&\leq \mathbb{E}[\|(W^{\perp}(s;t))\|\|y\|\|(W^{\perp}(s;t))\|] \\
&= \|y\|\mathbb{E}[\|(W^{\perp}(s;t))\|^2].
\end{aligned}$$

Altogether this means that

$$\mathbb{E}[W^{\perp}(s;t)W^{\|}(s;t)W^{\perp}(s;t)\,|\,\mathcal{F}_s] = \mathcal{L}(W^{\|}(s;t))$$

for some bounded operator $\mathcal{L} \in L(B)$. This shows that the independent increment process $(W(t))_{t\geq 0}$ is *not* a polynomial process in a non-commutative Banach algebra B. This is very different from the commutative case, where we recall from Proposition 21 that independent increment processes are polynomial processes.

Motivated by the above derivation, we may ask the question whether $(W(t))_{t\geq 0}$ is a *generalized* polynomial processes. However, this is not the case as can be seen by looking at $\mathbb{E}[W^5(t)\,|\,\mathcal{F}_s]$: Similar as the calculation above yields one term of the form

$$\begin{aligned}&\mathbb{E}[W^{\perp}(s;t)W^{\|}(s;t)W^{\perp}(s;t)W^{\|}(s;t)W^{\perp}(s;t)\,|\,\mathcal{F}_s]\\&\qquad=\mathbb{E}[W^{\perp}(s;t)yW^{\perp}(s;t)yW^{\perp}(s;t)]_{y=W^{\|}(s;t)}\end{aligned}$$

and the question is whether this expression can be written as $\mathcal{L}((W^{\|}(s;t))^2)$ for some (different than the above) $\mathcal{L}\in L(B)$. Let us assume that this is indeed the case, that is,

$$f(y):=\mathbb{E}[W^{\perp}(s;t)yW^{\perp}(s;t)yW^{\perp}(s;t)]=\mathcal{L}(y^2).$$

By Proposition 14 we know that

$$\begin{aligned}D^2f(y)(h_1,h_2)&=\mathbb{E}[W^{\perp}(s;t)h_1W^{\perp}(s;t)h_2W^{\perp}(s;t)]\\&\quad+\mathbb{E}[W^{\perp}(s;t)h_2W^{\perp}(s;t)h_1W^{\perp}(s;t)]\end{aligned}$$

and by Corollary 16 that

$$D^2(\mathcal{L}(y^2))(h_1,h_2)=\mathcal{L}(h_1h_2)+\mathcal{L}(h_2h_1).$$

If $f(y)=\mathcal{L}(y^2)$ then of course also their derivatives agree and

$$D^2f(y)(h_1,h_2)=D^2(\mathcal{L}(y^2))(h_1,h_2)$$

for every $h_1,h_2\in B$. To see that this can not be the case in general we look at the vector space $\mathbb{R}^{2\times 2}$ of 2×2-matrices equipped with a sub-multiplicative matrix norm and the usual matrix product. This space is well-known to be a non-commutative Banach algebra. Choose

$$h_1:=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad h_2:=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

First observe that $h_1\cdot h_2=h_2\cdot h_1=\mathbf{0}_2$ with $\mathbf{0}_2$ being the 2×2-matrix of zeros, then $D^2(\mathcal{L}(y^2))(h_1,h_2)=\mathcal{L}(\mathbf{0}_2)+\mathcal{L}(\mathbf{0}_2)=\mathbf{0}_2$ independent of the specification of $(W(t))_{t\geq 0}$. On the other hand, let now $(L_{ij}(t))_{t\geq 0}$ for $i,j=1,2$ be 4 independent copies of the real-valued Lévy processes $(L(t))_{t\geq 0}$ with finite moments of all orders. Then

$$W(t)=\begin{pmatrix}L_{11}(t)&L_{12}(t)\\L_{21}(t)&L_{22}(t)\end{pmatrix},$$

defines an independent increment process in the space of 2×2-matrices. It follows that

$$W^{\perp}(s;t)=\begin{pmatrix}\Delta_{s,t}L_{11}&\Delta_{s,t}L_{12}\\\Delta_{s,t}L_{21}&\Delta_{s,t}L_{22}\end{pmatrix}$$

where $\Delta_{s,t}L_{ij}(t)=L_{ij}(t)-L_{ij}(s)$ for $s\leq t$. We derive,

$$\begin{aligned}&D^2f(y)(h_1,h_2)\\=&\begin{pmatrix}2\mathbb{E}[\Delta_{s,t}L]^3&\mathbb{E}[\Delta_{s,t}L]^3+\mathbb{E}[(\Delta_{s,t}L)^2]\mathbb{E}[\Delta_{s,t}L]\\\mathbb{E}[\Delta_{s,t}L]^3+\mathbb{E}[(\Delta_{s,t}L)^2]\mathbb{E}[\Delta_{s,t}L]&2\mathbb{E}[\Delta_{s,t}L]^3\end{pmatrix}\neq\mathbf{0}_2\end{aligned}$$

whenever $\mathbb{E}[L(t)]\neq 0$. Choosing a real-valued Lévy process with mean unequal to zero yields a contradiction to $D^2(\mathcal{L}(y^2))(\bar{h}_1,\bar{h}_2)=\mathbf{0}_2$. So, in general independent increment processes fail to be even a generalized polynomial processes in a non-commutative Banach algebra. This shows that even for general Banach algebras one must introduce monomials in $M_k(B)$ as the structure preserving class to extend the notion of "polynomial processes" to infinite dimensions, and not merely polynomials nor generalized polynomials.

5. Applications

In this section we want to elaborate a bit more on some of the possible applications.

5.1. **Calculation of moments.** For multilinear processes in Hilbert space we can compute conditional moments. To this end, suppose B is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Define for $k \in \mathbb{N}$,

$$\mathcal{L}_{2k}(x_1, y_1, \dots, x_k, y_k) := \langle x_1, y_1 \rangle \cdots \langle x_k, y_k \rangle.$$

$\mathcal{L}_{2k}$ is a multilinear form, which is obviously bounded. We have

$$\mathcal{M}_{2k}(x) := \|x\|^{2k}$$

for any $x \in B$. Thus, if X is a multilinear process with respect to the family of B-valued random variables $(X(s;t))_{0 \le s \le t < \infty}$, then by Proposition 18,

$$\mathbb{E}[\|X(t)\|^{2k}] = \mathbb{E}[\mathcal{M}_{2k}(X(t))] = \sum_{j=1}^{2k} \overline{\mathcal{M}}_j(X(0;t)),$$

for a family jth-order monomials $\overline{\mathcal{M}}_j : B \to \mathbb{K}$, $j = 1, \dots, 2k$. So, we can compute even moments of the norm of X in terms of multilinear forms operating on $X(0;t)$ of order at most $2k$, where $X(0;t)$ is $\mathcal{F}_0$-measurable.

For the odd moments, we note that for $k = 0, 1, 2, \dots$, it obviously holds that $2k + 1 = \alpha(k) \times (2k + 2)$ for $\alpha(k) = (2k + 1)/(2k + 2) \in (1/2, 1)$. One has that (see Applebaum [2], page 80)

$$u^{\alpha(k)} = \frac{\alpha(k)}{\Gamma(1 - \alpha(k))} \int_0^\infty (1 - e^{-ux}) x^{-\alpha(k)-1} dx$$

Thus, we find the representation

$$\mathbb{E}[\|X(t)\|^{2k+1}] = \frac{\alpha(k)}{\Gamma(1 - \alpha(k))} \int_0^\infty (1 - \mathbb{E}[\exp(-x\|X(t)\|^{2k+2})]) x^{-1-\alpha(k)} dx$$

Doing a series representation of the exponential function inside the integral on the right hand side, we find that

$$\mathbb{E}[\|X(t)\|^{2k+1}] = \frac{\alpha(k)}{\Gamma(1 - \alpha(k))} \int_0^\infty \sum_{\ell=1}^\infty \frac{(-1)^\ell}{\ell!} x^{-\alpha(k)-1+\ell} \mathbb{E}[\|X(t)\|^{2\ell(k+1)}] dx$$

Thus, we can use the multilinear property of a process $(X(t))_{t \ge 0}$ to compute an integral of an infinite series of even moments to find any odd moment.

5.2. **Applications to commodity markets.** A forward contract is a financial arrangment where the seller promises to deliver an underlying commodity (like for example oil, coffee, aluminium or power) at an agreed price at some future time point. Entering such a contract at time $t \ge 0$ where delivery takes place at time $t + x, x \ge 0$ in the future, we denote the agreed *forward price* by $f(t, x)$. It is known (see Benth and Krühner [4]) that $t \mapsto f(t, \cdot)$ can be interpreted as a stochastic process with values in some Hilbert space of continuous functions on $\mathbb{R}_+$, solving (mildly) the stochastic partial differential equation (23) with $\mathcal{A} = \partial/\partial x$. This model is a special class of a more general stochastic partial differential equation dynamics, belonging to the Heath-Jarrow-Morton modelling paradigm (see e.g. Filipović [12], Geman [15], Carmona and Tehranchi [8] for more on this, including the case of forward rates in fixed-income markets).

Following Benth and Krühner [4], a natural state space of the forward price curves is the Filipović space (see Filipović [12]). The Filipović space H_w is defined for an increasing, continuous function $w : \mathbb{R}_+ \to [1, \infty)$ with $w(0) = 1$ to be the set of functions

$$H_w := \left\{ g \in AC(\mathbb{R}_+, \mathbb{R}) : \int_0^\infty w(x) g'(x)^2 dx < \infty \right\}, \tag{25}$$

where $AC(\mathbb{R}_+, \mathbb{R})$ denotes the set of absolutely continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. The scalar product $\langle g_1, g_2 \rangle := g_1(0) g_2(0) + \int_0^\infty w(x) g_1'(x) g_2'(x) dx$ for $g_1, g_2 \in H_w$ makes H_w a separable Hilbert space with norm $\|g\|_w^2 = |\langle g, g \rangle|$. As already observed in Benth and Krühner [4] assuming $w^{-1} \in L^1(\mathbb{R}_+)$, the pointwise multiplication defines an algebra on H_w and with the new norm $|\cdot|_{w,c} := c|\cdot|_w$, where $c = \sqrt{1 + 8(1 + \int_0^\infty w^{-1}(x)\, dx)}$ the space H_w is actually a commutative Banach algebra.

On the commutative Banach algebra $B = H_w$, we have that the shift operator $\mathcal{S}_t g := g(\cdot + t)$ defines is a C_0-semigroup being a homomorphism. Moreover, the generator of $(\mathcal{S}_t)_{t \geq 0}$ is the derivative operator $\partial / \partial x$. Thus, in light of the discussion in Section 4, the forward curve dynamics $(f(t, \cdot))_{t \geq 0}$ is given by the stochastic convolution process (22), and recalling (24), will become a generalized polynomial process on H_w with respect to $(f(s, \cdot))_{0 \leq s \leq t < \infty}$. In representation (24), we will have $X^\perp(s; t) := \int_s^t \mathcal{S}_{t-u} dW(u)$ with $\mathcal{S}_t$ being the shift operator.

Let us give an application where generalized polynomial property comes in handy. In commodity markets, options on forwards are popular risk management products. Let us consider a general calendar spread type of option, defined as follows. At time t, the holder can exercise the option yielding a payment

$$h(f(t, x_1), f(t, x_2), \ldots, f(t, x_n))$$

with $0 \leq x_1 < x_2 < \cdots < x_n$. The function $h : \mathbb{R}^n \to \mathbb{R}$ is a measurable function describing the payment. Typical cases in the markets are $h(x) = \max(x - K, 0)$ for a standard call option (with $n = 1$), or $h(x, y) = \max(x - y, 0)$ for a calendar spread option between two delivery times ($n = 2$). Let δ_x denote the evaluation map at $x \geq 0$. It is shown in Filipović [11] that δ_x is a bounded linear functional on H_w. Thus,

$$h(f(t, x_1), f(t, x_2), \ldots, f(t, x_n)) = h(\delta_{x_1} f(t), \ldots, \delta_{x_n} f(t)),$$

and the price of the calendar option at time $s \leq t$ is given by

$$P(s, t) = \mathbb{E}\left[h\left(\delta_{x_1} f(t), \ldots, \delta_{x_n} f(t) \right) \mid \mathcal{F}_s \right] \tag{26}$$

assuming zero risk-free interest rate (see Benth et al. [6]). Assume now that there exists a polynomial representation of h,

$$h(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n = 0}^\infty h_{i_1 i_2 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n} \tag{27}$$

and that

$$P(s,t) = \sum_{i_1,\dots,i_n=0}^{\infty} h_{i_1\cdots i_n} \mathbb{E}\left[(\delta_{x_1} f(t))^{i_1} \cdots (\delta_{x_n} f(t))^{i_n} \,|\, \mathcal{F}_s\right]. \tag{28}$$

Introduce now the functional $\mathcal{L}_{x_1,\dots,x_n} : H_w^{\times n} \to \mathbb{R}$ defined as $(g_1,\dots,g_n) \ni H_w^{\times n} \mapsto \delta_{x_1} g_1 \cdot \delta_{x_2} g_2 \cdots \delta_{x_n} g_n \in \mathbb{R}$. $\mathcal{L}_{x_1,\dots,x_n}$ is directly verified as continuous when equipping $H_w^{\times n}$ with the 2-norm. As δ_x is a linear functional on H_w, we can apply a modified version of Lemma 1 to show that

$$\delta_{x_j} \mathbb{E}[f(t)^{i_j} \,|\, \mathcal{F}_s] = \mathbb{E}[\delta_{x_j}(f(t)^{i_j}) \,|\, \mathcal{F}_s] = \mathbb{E}[(\delta_{x_j} f(t))^{i_j} \,|\, \mathcal{F}_s]$$

It follows from (24) that

$$\begin{aligned}
P(s,t) &= \sum_{i_1,\dots,i_n=0}^{\infty} h_{i_1\cdots i_n} \mathcal{L}_{x_1,\dots,x_n}\left(\mathbb{E}\left[(f(t))^{i_1} \,|\, \mathcal{F}_s\right], \dots, \mathbb{E}\left[(f(t))^{i_n} \,|\, \mathcal{F}_s\right]\right) \\
&= \sum_{i_1,\dots,i_n=0}^{\infty} h_{i_1\cdots i_n} \mathcal{L}_{x_1,\dots,x_n}\left(\sum_{j=0}^{i_1} \binom{i_1}{j} \mathbb{E}\left[(X^{\perp}(s;t))^{i_1-j}\right] \mathcal{S}_{t-s} f^j(s), \dots,\right. \\
&\qquad \left.\dots, \sum_{j=0}^{i_n} \binom{i_n}{j} \mathbb{E}\left[(X^{\perp}(s;t))^{i_n-j}\right] \mathcal{S}_{t-s} f^j(s)\right) \\
&= \sum_{i_1,\dots,i_n=0}^{\infty} h_{i_1\cdots i_n} \left(\sum_{j=0}^{i_1} \binom{i_1}{j} \delta_{x_1}(\mathbb{E}\left[(X^{\perp}(s;t))^{i_1-j}\right]) \delta_{x_1+t-s} f^j(s)\right) \cdots \\
&\qquad \cdots \left(\sum_{j=0}^{i_n} \binom{i_n}{j} \delta_{x_n}(\mathbb{E}\left[(X^{\perp}(s;t))^{i_n-j}\right]) \delta_{x_n+t-s} f^j(s)\right),
\end{aligned}$$

where we have used that $\delta_x(g \cdot h) = \delta_x(g)\delta_x(h)$ for any $g,h \in H_w$ and $\delta_x \mathcal{S}_t = \delta_{x+t}$ for every $x,t \geq 0$.

5.3. **Some other choices of Banach spaces.** A canonical example of a separable Banach space is the space $C([0,1])$ of real-valued continuous functions $f : [0,1] \to \mathbb{R}$ equipped with pointwise product and uniform norm $|f|_\infty := \sup_{x\in[0,1]} |f(x)|$. This is also a commutative Banach algebra, and we notice that it is the path space of Brownian motion.

Another classical separable Banach space is $L^p(\mathbb{R}^d)$, the space of p-integrable functions on $\mathbb{R}^d$ for $p,d \in \mathbb{N}$. As is well-known, $L^2(\mathbb{R}^d)$ is a Hilbert space and also possible state-space for Gaussian random fields. One can define a multiplication for $f,g \in L^1(\mathbb{R}^d)$ by the convolution product, i.e.

$$f * g(x) = \int_{\mathbb{R}^d} f(y-x) g(y) dy. \tag{29}$$

This turns $L^1(\mathbb{R}^d)$ into a commutative Banach algebra. A possible application could be stochastic processes taking values in $L^1(\mathbb{R})$ being probability densities, e.g. being non-negative integrable functions with unit mass.

Another classical Banach algebra is the space of bounded linear operators $B = L(C)$ on the Banach space C, equipped with the operator norm. The space B forms a non-commutative Banach algebra under the standard operator product. If C is separable Hilbert space, one can consider the subspace of Hilbert-Schmidt operators

$L_{\mathrm{HS}}(C)$, which becomes a separable Hilbert space, however, not an algebra. In Benth, Rüdiger and Süss [5] and Benth and Simonsen [7] positive-definite Hilbert-Schmidt-valued Ornstein-Uhlenbeck processes have been defined and studied in the context of stochastic volatility models in infinite dimensions. These volatility models become multilinear processes.

Let $(E, \mathcal{E})$ be a measurable space, and denote by $\mathbb{M}(E)$ the space of all finite signed measures. Equip $\mathbb{M}(E)$ with the total variation norm, $\|\nu\|_{\mathrm{TV}} := |\nu|(E)$ for $\nu \in \mathbb{M}(E)$. It is known that $(\mathbb{M}(E), \|\cdot\|_{\mathrm{TV}})$ is a Banach space. Define the convolution product of measures as

$$\nu * \mu(A) = \int_{E\times E} \mathbf{1}_A(x+y)\nu(dx)\mu(dy) = \int_E \nu(A-y)\mu(dy)$$

for $\nu, \mu \in \mathbb{M}(E)$ and $A \in \mathcal{E}$. Since $\|\nu * \mu\|_{\mathrm{TV}} \leq \|\nu\|_{\mathrm{TV}}\|\mu\|_{\mathrm{TV}}$, $(\mathbb{M}(E), \|\cdot\|_{\mathrm{TV}}, *)$ is a Banach algebra which obviously is commutative. A polynomial $p_k \in \mathrm{Pol}_k(\mathbb{M}(E))$ will be of the form $p_k(\mu) = \sum_{n=0}^k \nu_n * \mu^{*n}$ for $(\nu_n)_{n=0}^k \subset \mathbb{M}(E)$. These polynomials are built up from the monomials μ^{*n}. Cuchiero, Larsson and Svaluto-Ferro [10] define polynomial processes on $\mathbb{M}(E)$ introducing *monomials* as follows: Let $g : E^k \to \mathbb{R}$ be a continuous symmetric function. A monomial of degree $k \in \mathbb{N}$ is defined as

$$\mathbb{M}(E) \ni \nu \mapsto \langle g, \nu^k \rangle := \int_{E^k} g(x_1, \dots, x_k)\nu(dx_1)\cdots\nu(dx_k)$$

We notice that for any $A \in \mathcal{E}$, we have that

$$\nu^{*k}(A) = \int_{E^k} \mathbb{I}_A(x_1 + \cdots + x_k)\nu(dx_1)\cdots\nu(dx_k) = \langle \mathbb{I}_A(x_1 + \cdots + x_k), \nu^k \rangle.$$

Although the function $(x_1, \dots, x_k) \mapsto \mathbb{I}_A(x_1 + \cdots + x_k)$ is obviously not continuous, it is a bounded measurable symmetric function which is linking our definition of polynomial processes on $\mathbb{M}(E)$ to the one of Cuchiero, Larsson and Svaluto-Ferro [10].

References

[1] D. Ackerer, D. Filipović and S. Pulido (2018). The Jacobi stochastic volatility model. *Finance Stoch.* 22: 667–700.

[2] D. Applebaum (2004). *Lévy Processes and Stochastic Calculus*, Cambridge University Press

[3] D. Applebaum (2007). Lévy processes and stochastic integrals in Banach spaces. *Prob. Math. Stat.* 27: 75–88.

[4] F. E. Benth and P. Krühner (2014). Representation of infinite-dimensional forward price models in commodity markets. *Commun. Math. Stat.* 2:47–106

[5] F. E. Benth, B. Rüdiger and A. Süss (2018). Ornstein-Uhlenbeck processes in Hilbert space with non-Gaussian stochastic volatility. *Stoch. Proc. Applic.*, 128: 461–486.

[6] F. E. Benth, J. Šaltytė Benth, and S. Koekebakker (2008). *Stochastic Modelling of Electricity and Related Markets*, World Scientific, Singapore.

[7] F. E. Benth and I. C. Simonsen (2018+). The Heston stochastic volatility model in Hilbert space. To appear in *Stoch. Analysis Appl.*.

[8] R. Carmona, and M. Tehranchi (2006). *Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective*, Springer Verlag

[9] C. Cuchiero, M. Keller-Ressel and J. Teichmann (2012). Polynomial processes and their applications to mathematical finance. *Finance Stoch.* 16:711–740

[10] C. Cuchiero, M. Larsson and S. Svaluto-Ferro (2018). Measure valued polynomial diffusions. *Working paper.*

[11] N. Dinculeanu (2000). *Vector Integration and Stochastic Integration in Banach Spaces*. John Wiley & Sons.

[12] D. Filipović (2001). *Consistency Problems for Heath-Jarrow-Morton Interest Rate Models.* Lecture Notes in Mathematics, vol. 1760. Springer, Berlin.

[13] D. Filipović and M. Larsson (2016). Polynomial diffusions and applications in finance. *Finance Stoch.* 4: 931–972

[14] J.L. Forman and M. Sørensen (2008). The Pearson diffusions: a class of statistically tractable diffusion processes. *Scand. J. Stat.* 35: 438–465

[15] H. Geman (2005). *Commodities and Commodity Derivatives.* John Wiley & Sons, Chichester.

[16] J. van Neerven (2010). *Stochastic Evolution Equations*, ISEM Lecture Notes 2007/8. Downloaded from http://fa.its.tudelft.nl/ neerven/publications/notes/ISEM.pdf in May 2018.

[17] Peszat and Zabczyk (2007). *Stochastic Equations in Infinite Dimensions.* Cambridge University Press

[18] J. Walsh (1986), An introduction to stochastic partial differential equations. In R. Carmona, H. Kesten & J. Walsh (eds.), *Lecture Notes in Mathematics*, 1180, Ecole d'Eté de Probabilités de Saint-Flour XIV (1984), Springer Verlag.

Fred Espen Benth, University of Oslo, Department of Mathematics, P.O. Box 1053, Blindern, N–0316 Oslo, Norway

E-mail address: `fredb@math.uio.no`

Nils Detering, Department of Statistics and Applied Probability, CA 93106 Santa Barbara, USA

E-mail address: `detering@pstat.ucsb.edu`

Paul Krühner, Institute for Financial and Actuarial Mathematics, The University of Liverpool, The Mathematical Sciences Building, Liverpool L69 7ZL, UK

E-mail address: `paul.eisenberg@liverpool.ac.uk`